MATH 20D Spring 2023 Lecture 8.

More on Linear Independence, and General Solutions to 2nd order ODE's

Outline

Announcements

- Mistake in Friday and Mondays lecture. The volume of water in the tank should have been computed as

$$V(t) = \begin{cases} 180, & 0 \leqslant t \leqslant 10 \\ 180 - (t - 10), & t > 10. \end{cases}$$

See Zulip Lecture Q & A stream for a detailed discussion.

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- Midterm 1 is approaching (next Wednesday WLH 2005 during lecture)
- Homework 3 is posted, please use most recent version updated around 12:30pm today.
- Review Sheet will come out tomorrow with solutions posted over the weekend. Unfortunately I will not be able to release solutions to homework until the late due date elapses on the Saturday following the midterm.

Contents

Definition

We say that functions $y_1, y_2 : \mathbb{R} \to \mathbb{R}$ are ***linearly dependent*** *if there exists a constant $\alpha \in \mathbb{R}$ such that*

$$y_1(t) = \alpha y_2(t)$$

for all $t \in \mathbb{R}$. If not, we say that y_1 and y_2 are ***linearly independent***

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Lemma

Fix functions $y_1, y_2 : \mathbb{R} \to \mathbb{R}$. If y_1 and y_2 are linearly dependent then

$$P = \begin{pmatrix} y_1(0) \\ y_1'(0) \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} y_2(0) \\ y_2'(0) \end{pmatrix}$$

represent vectors in $\mathbb{R}^2$ lying on the same stright line.

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Show that the following pairs of functions are linearly independent;

(a) $y_1(x) = \cos(x)$, $y_2(x) = \sin(x)$

(b) $y_1(x) = e^{r_1 x}$, $y_2(x) = e^{r_2 x}$ provided $r_1 \neq r_2$.

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Let $a \neq 0$, b, and c be constants and consider the ODE

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(a) *Then (1) admits a pair of linearly independent solutions.*

(b) *Suppose y_1 and y_2 are* ***linearly independent*** *solutions to (1). Then*

$$y\colon \mathbb{R} \to \mathbb{R}, \quad y(t) = C_1 y_1(t) + C_2 y_2(t)$$

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If $b^2 - 4ac > 0$,

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are linearly independent solution to (1).

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then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are linearly independent solution to (1). Hence if $b^2 - 4ac > 0$ then (1) admits a general solution of the form

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

Example

Consider the ODE

$$y'' + 6y' + 3y = 0 \tag{2}$$

- Determine constants r_1 and r_2 such that the ODE above admits a general solution of the form $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.
- Solve (2) subject to the initial conditions $y(0) = 0$ and $y'(0) = 1$.

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- The **auxiliary equation** $r^2 + 6r + 3 = 0$ has distinct real roots

$$r_1 = -3 + \sqrt{6} \qquad \text{and} \qquad r_2 = -3 - \sqrt{6}.$$

Therefore $y(t) = C_1 e^{(-3+\sqrt{6})t} + C_2 e^{(-3-\sqrt{6})t}$ is a general solution.

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  Solving by elimination (or substitution) $C_1 = \sqrt{6}/12$ and $C_2 = -\sqrt{6}/12$.
- Obtain solution $y(t) = \frac{\sqrt{6}}{12}\left(e^{(-3+\sqrt{6})t} - e^{(-3-\sqrt{6})t}\right)$.

Question

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Theorem

Suppose $b^2-4ac=0$ and let $r=-b/2a$. Then

$$y_1(t)=e^{rt} \qquad \text{and} \qquad y_2(t)=te^{rt}$$

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Example

(a) Write down a general solutions to the ODE $y'' - 4y' + 4y = 0$.

(b) Solve the IVP

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

The Case of Conjugate Complex Roots I

- Suppose $b^2 - 4ac < 0$. Then $ar^2 + br + c = 0$ has solutions

$$r_1 = \frac{1}{2a}(-b + i\sqrt{4ac - b^2}) \qquad \text{and} \qquad r_2 = \frac{1}{2a}(-b - i\sqrt{4ac - b^2})$$

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Example

(a) Write down a general solution to the ODE $y'' + 2y' + 4y = 0$.
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where $\alpha \pm i\beta$ are the roots to the equation $ar^2 + br + c = 0$.

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where $\alpha \pm i\beta$ *are the roots to the equation* $ar^2 + br + c = 0$*. Then (4) can be rewritten in the form*

$$y(t) = Ae^{\alpha t}\sin(\beta t + \phi)$$

where $A = \sqrt{C_1^2 + C_2^2}$ *and* $\phi \in [0, 2\pi)$ *satisfies* $C_1 = A\sin(\phi)$ *and* $C_2 = A\cos(\phi)$*.*

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Example

(a) Solve the IVP $\frac{1}{8}y''(t) + 16y(t) = 0$, $y(0) = 1/2$, $y'(0) = -\sqrt{2}$.
(b) Rewrite your solution to (a) in the form $y(t) = Ae^{\alpha t}\sin(\beta t + \phi)$.